Mirror Symmetry in the Few Anyon Spectra in a Harmonic Oscillator Potential

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Abstract

We find that the energy spectra of four and five anyons in a harmonic potential exhibit some mirror symmetric (reflection symmetric about the semionic statistics point $\theta = \pi/2$) features analogous to the mirror symmetry in the two and three anyon spectra. However, since the $\ell = 0$ sector remains non-mirror symmetric, the fourth and fifth virial coeffients do not reflect this symmetry.

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The multi-anyon problem continues to remain interesting as it defies both analytic and numerical attempts to solve it. As has been often reiterated in the literature [1,2], the difficulty stems from the inability to write multi-anyon wave-functions as simple products of single anyon wave-functions. Hence, even simple quantum mechanical problems involving more than two anyons remain insoluble.

The two standard routes that have been followed to tackle the problem have been either to start from the high density limit or from the low density limit. In the high density limit [3], the anyons are stripped of their fluxes, which get spread out and the mean field problem involves fermions or bosons moving in a uniform magnetic field. This is the approach that led to anyon superconductivity. Standard improvements in the mean field theory by incorporation of higher order corrections have been made; however, in the absence of a small expansion parameter, the validity of these approximations remains unclear.

In the low density limit [4], the strategy has been to solve few anyon problems. For more than two anyons, the problem cannot be solved exactly. However, for free anyons, anyons in a magnetic field and anyons in a harmonic potential, several exact wave-functions of the Hamiltonian can be constructed [5]. But since these wave-functions do not form a complete set, they cannot be used to calculate statistical quantities like the virial coefficients. For this, perturbation theory [6] about the bosonic or fermionic limit has been tried using the statistical parameter as the expansion parameter. Also, the lowest lying eigenstates of the Hamiltonian have been obtained numerically for the three [7] and four [8] anyon problems. These approaches gave some insight into the virial coefficients and hence the statistical mechanics of a gas of anyons.

More recently, a useful approach in the few anyon system was pioneered by Sen [9], who showed that by studying the symmetries of the Hamiltonian of the few anyon system, some exact statements could be proven about the spectrum. He showed that the three anyon spectrum in a harmonic potential is (almost) mirror symmetric about the semionic point ($\theta = \pi/2$) by constructing a fermionic operator Q_3 (a quadratic polynomial) that commuted with the Hamiltonian. More interestingly, by using this symmetry, he was able to show that the third virial coefficient is mirror-symmetric about $\theta = \pi/2$. However, so far, no exact statements have been made for more than three anyons.

With the view that any exact statement that can be made about the many anyon spectrum is a useful excercise, in this paper, we study four and five anyon spectra in a harmonic potential in some detail. In analogy with Ref. [9], we construct fermionic operators. For the four anyon case, we construct four fermionic operators Q_4^i, which are cubic polynomials. These operators do not commute with the Hamiltonian - rather, they act as raising and lowering operators. We show that all states (other than the $\ell = 0$ states and those linear states that are annihilated by Q_4^i) have a 'skewed' mirror symmetry about $\theta = \pi/2$ - i.e., a state with energy E and angular momentum ℓ is paired by the Q_4^i with states with energy $E \pm \omega$ and angular momentum $\ell \pm 1$. For the five anyon case, however, a unique fermionic operator Q_5 can be constructed, (a fourth order polynomial), which commutes with the Hamiltonian and whose action can be used to demonstrate a genuine mirror symmetry of the spectrum for all states (other than $\ell = 0$ states and those linear states annihilated by Q_5). However, the absence of the mirror symmetry for the $\ell = 0$ states (which includes non-exactly solved states) implies that this is not sufficient to make any exact statement about the fifth virial coefficient.

We start by considering the problem of N anyons in a harmonic potential. This is a generic problem, since results for anyons in a magnetic field as well as free anyons can be derived from these results. We shall work in the bosonic gauge where the wave-function is bosonic and the information about the fractional statistics is incorporated in the Hamiltonian. The transformation to the anyon gauge is simply achieved by changing the Hamiltonian to a free Hamiltonian and multiplying the wave-functions by the phase $\Delta/|\Delta|$ where

$$\Delta = \Pi^N_{n<m}(z_n - z_m) \tag{1}$$

and $z_n = (x_n + iy_n)/\sqrt{2}$ are complex coordinates on a plane. The Hamiltonian in the bosonic gauge is given by

$$\mathcal{H} = -\sum_{n=1}^N [(\frac{\partial}{\partial z_n} + s_n)(\frac{\partial}{\partial z_n^*} - s_n^*) + \omega^2 z_n z_n^*] \tag{2}$$

where we have set the mass of the particles $m = 1$ and the gauge potentials are given by

$$s_n = \frac{\alpha}{2} \sum_{n\neq m}^N \frac{1}{z_n - z_m} = \frac{\alpha}{2} \frac{\partial}{\partial z_n} \log \Delta. \tag{3}$$

Here, α is the statistics parameter , ($= \theta/\pi$) and ranges from $\alpha = 0$ to $\alpha = 1$. Since the gauge potentials are functions only of the relative coordinates, it is more convenient to use Jacobi coordinates defined by

$$u_0 = \frac{1}{\sqrt{N}} \sum_{n=1}^N z_n \equiv \alpha_{0m} z_m,$$

$$u_i = \frac{1}{\sqrt{i(i+1)}}(z_1 + z_2 + \cdots - i\ z_{i+1}) \equiv \alpha_{im} z_m \tag{4}$$

with $i = 1, \cdots, N-1$. In terms of these coordinates, the Hamiltonian neatly splits into a CM part H_{CM} which is a free oscillator and a relative part H which is given by

$$H = -\sum_{i=1}^{N-1} [(\frac{\partial}{\partial u_i} + v_i)(\frac{\partial}{\partial u_i^*} - v_i^*) + \omega^2 u_i u_i^*] \tag{5}$$

with the gauge potentials v_i being given by

$$v_i = \frac{\alpha}{2} \frac{\partial}{\partial u_i} \log D, i = 1, \cdots, N-1 \tag{6}$$

and $D \equiv D(u_i)$ is the transformed form of Δ. This is easily proved using the orthogonality of the transformation in Eq. 4 and the analyticity of s_n and v_i. Note that D is independent of u_0, since it only depends on relative coordinates. (A slightly different form of v_i is used in Ref. [9].)

Let us now introduce the ladder operators $a_i, a_i^\dagger, b_i$ and $b_i^\dagger$ -

$$a_i = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_i} + v_i + \omega u_i^*), \quad b_i = \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial u_i^*} + v_i^* - \omega u_i),$$

$$a_i^\dagger = \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial u_i^*} + v_i^* + \omega u_i), \quad b_i^\dagger = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_i^*} + v_i^* - \omega u_i^*), \quad (7)$$

for $i = 1, \cdots, N-1$. The a_i and $a_i^\dagger$ commute with b_i and $b_i^\dagger$ and

$$[a_i, a_j^\dagger] = \omega \delta_{ij}, \quad [b_i, b_j^\dagger] = \omega \delta_{ij}. \quad (8)$$

The Hamiltonian and the relative angular momentum operators $L = u_i \frac{\partial}{\partial u_i} - u_i^* \frac{\partial}{\partial u_i^*}$ can be written in terms of these operators as

$$H = \sum_{i=1}^{N-1} (a_i^\dagger a_i + b_i^\dagger b_i) + (N-1)\omega$$

$$\text{and} \quad \omega L = \sum_{i=1}^{N-1} (a_i^\dagger a_i - b_i^\dagger b_i) - \frac{N(N-1)}{2}\alpha. \quad (9)$$

Although the Hamiltonian may look trivially solvable, note that the ladder operators are not the conventional ones. Firstly, they transform as an $N-1$-dimensional irreducible representation (IR) of the permutation group S_N - *i.e.*, if $|\psi\rangle$ is a bosonic state, then $a_i^\dagger|\psi\rangle$ or $b_i^\dagger|\psi\rangle$ are not bosonic or fermionic , but transform as an $N-1$-dimensional IR of S_N. Also, the wavefunctions of the Hamiltonian must vanish as $|z_i - z_j|^\alpha$ or faster as z_i approaches z_j. This is the hard-core constraint on the bosons which can be deduced from the statistics of the anyons in the anyon gauge. Such wave-functions are called physical or non-singular. But there is no guarantee that $a_i^\dagger$ and $b_i^\dagger$ acting on physical wavefunctions will only lead to physical wavefunctions. They could lead to singular wave-functions which must be rejected.

However, Sen [9] showed that polynomials of these ladder operators could be formed, which transform as bosons (Q_B) or fermions (Q_F). He showed that of ten possible quadratic bosonic operators, $K_+ = 2a_i^\dagger b_i^\dagger$ and $K_- = 2a_i b_i$ are 'good' operators - *i.e.*, they do not produce unphysical states when acting on physical states - and along with $K_3 = H$, they form an $SO(2,1)$ algebra [11] given by

$$[H, K_\pm] = \pm 2\omega K_\pm. \quad (10)$$

This shows that the spectrum gets organised in terms of $SO(2,1)$ families, with members within each family differing in energy from each other by 2ω.

For the three anyon problem, he was also able to prove a much more interesting result by constructing a 'good' fermionic operator

$$\widetilde{Q}_3 = a_1^\dagger b_2 - a_2^\dagger b_1. \quad (11)$$

Since this operator is fermionic, it turns bosons into fermions and vice-versa - *i.e.*, it changes the statistics parameter α to $\alpha + 1$. A parity operator P can also be defined which maps $1+\alpha$ to $1-\alpha$ and hence the combined operator $Q_3 = \widetilde{Q}_3 P$ maps bosonic states at statistics parameter α to bosonic states at statistics $1-\alpha$, (which are equivalent to fermionic states at

statistics $-\alpha$). Since, this operator commuted with the Hamiltonian, he essentially proved that the three anyon spectrum is mirror symmetric about the semionic point $\alpha = 1/2$ (except for some of the linear exactly solved states, which were annihilated by Q_3). He further used this symmetry to prove that the third virial coefficient is exactly mirror symmetric.

We now try to extend these results to the case when $N = 4$. Our aim is to construct fermionic or bosonic operators that enable us to make some exact statements about the spectrum. Bosonic bilinears (quadratic operators) can be constructed as before and the $SO(2,1)$ symmetry can be extended to the $N = 4$ case. But unlike the three anyon case, no fermionic bilinear operator can be constructed. However, consider the following trilinear operators -

$$
\begin{aligned}
Q_4^1 &= \epsilon_{ijk} a_i^\dagger b_j b_k^\dagger \\
Q_4^2 &= \epsilon_{ijk} a_i a_j^\dagger b_k^\dagger \\
Q_4^3 &= \epsilon_{ijk} a_i a_j^\dagger b_k \\
Q_4^4 &= \epsilon_{ijk} a_i b_j b_k^\dagger.
\end{aligned}
\tag{12}
$$

By construction, these operators are anti-symmetric under $u_i \leftrightarrow u_j$. But they can also be proven to be antisymmetric under $z_m \leftrightarrow z_n$ in the following way. As far as the orthogonal transformation in Eq.4 is concerned, the coordinates u_i and u_i^* and the derivatives $\frac{\partial}{\partial u_i}$ and $\frac{\partial}{\partial u_i^*}$ transform with the same coefficients, -$i.e.$,

$$
\begin{aligned}
u_i &\rightarrow \alpha_{im} z_m \\
u_i^* &\rightarrow \alpha_{im} z_m^* \\
\frac{\partial}{\partial u_i} &\rightarrow \alpha_{im} \frac{\partial}{\partial z_m} \\
\frac{\partial}{\partial u_i^*} &\rightarrow \alpha_{im} \frac{\partial}{\partial z_m^*},
\end{aligned}
\tag{13}
$$

where α_{im} are defined in Eq.4. Moreover, the v_i and v_i^* also transform with the same coefficients because

$$
v_i = \frac{\alpha}{2} \frac{\partial}{\partial u_i} \log D = \frac{\alpha}{2D} \frac{\partial}{\partial u_i} D \rightarrow \frac{\alpha}{2\Delta} \alpha_{im} \frac{\partial}{\partial z_m} \Delta
$$

$$
\text{and} \quad v_i^* = \frac{\alpha}{2} \frac{\partial}{\partial u_i^*} \log D^* = \frac{\alpha}{2D^*} \frac{\partial}{\partial u_i^*} D^* \rightarrow \frac{\alpha}{2\Delta^*} \alpha_{im} \frac{\partial}{\partial z_m^*} \Delta^*.
\tag{14}
$$

This implies that we can now have new oscillators a_m and b_m defined by

$$
\begin{aligned}
a_i &= \alpha_{im} a_m \\
b_i &= \alpha_{im} b_m \\
a_i^\dagger &= \alpha_{im} a_m^\dagger \\
b_i^\dagger &= \alpha_{im} b_m^\dagger,
\end{aligned}
\tag{15}
$$

where $m = 1, ..4$. In terms of these oscillators, the Q_4^i can be written as

$$
Q_4^i = \epsilon_{jkl} \alpha_{jm} \alpha_{kn} \alpha_{lo} O_4^i =
\begin{vmatrix}
\alpha_{1m} & \alpha_{2m} & \alpha_{3m} \\
\alpha_{1n} & \alpha_{2n} & \alpha_{3n} \\
\alpha_{1o} & \alpha_{2o} & \alpha_{3o}
\end{vmatrix}
O_4^i \equiv \beta_{mno} O_4^i
\tag{16}
$$

where $O_4^i = a_m^\dagger b_n b_o^\dagger, a_m a_n^\dagger b_o^\dagger, a_m a_n^\dagger b_o$ and $a_n b_m b_o^\dagger$ respectively for $i = 1, \cdots 4$. Using the coefficients defined in Eq.4, it is easy to check that the determinant β_{mno} is completely antisymmetric in m, n and o and in fact, can be written as $\beta_{mno} = \epsilon_{mno}/2$. This proves that the operators Q_4^i are completely antisymmetric under $z_m \leftrightarrow z_n$. (A similar proof can be constructed for the antisymmetry of Q_3 using $\epsilon_{ij}\alpha_{im}\alpha_{jn} = \alpha_{mi}^T \epsilon_{ij}\alpha_{jn} = \epsilon_{mn}/\sqrt{3}$. This is somewhat different from the proof constructed by Sen [9].) Furthermore, these operators act as raising and lowering operators of the Hamiltonian and angular momentum -

$$\left[H, Q_4^i\right] = \omega Q_4^i, \quad i = 1, 2$$
$$\left[H, Q_4^i\right] = -\omega Q_4^i, \quad i = 3, 4$$
$$\left[L, Q_4^i\right] = Q_4^i, \quad , i = 1, 3$$
$$\left[L, Q_4^i\right] = -Q_4^i, \quad , i = 2, 4. \tag{17}$$

Are these 'good' operators? To answer this question, notice that we only have to examine the singularity as $z_1 \leftrightarrow z_2$, (i.e., $u_1 \to 0$), since the operators have already been shown to be antisymmetric. As $u_1 \to 0$, in any anyonic theory, physical wavefunctions can vanish as

$$i) \quad (u_1 u_1^*)^{|\alpha|/2}, \quad \ell = 0$$
$$ii) \quad (u_1 u_1^*)^{\alpha/2} u_1^\ell \quad \ell \geq 2 \quad (\ell = \text{even})$$
$$iii) \quad (u_1 u_1^*)^{-\alpha/2} u_1^{*\ell} \quad \ell \geq 2 \quad (\ell = \text{even})$$
$$iv) \quad (u_1 u_1^*)^{\alpha/2} u_1^\ell + (u_1 u_1^*)^{-\alpha/2} u_1^{*\ell} \quad \ell \geq 2 \quad (\ell = \text{even}), \tag{18}$$

where, in case iv), we mean that the two terms can be multiplied by different non-vanishing functions of the remaining u_i. For $0 < \alpha < 1$, it is easy to see that $a_1^\dagger$ and $b_1 (\simeq \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_1^*} - v_1^*)$ when $u_1 \to 0$) never produce any singularities when acting on any of the wave-functions (i) through (iv). But single powers of a_1 and $b_1^\dagger$ produce singular wave-functions when they act on the $\ell = 0$ wave-functions in (i). Since, all the Q_4^i contain either the term a_1 or $b_1^\dagger$, they are not 'good' operators per se; however, they produce singularities only when they act on the $\ell = 0$ wave-functions.

We can also find the states on which the action of the operators Q_4^i gives zero as follows. Any state annihilated by a_i will be annihilated by the sum $2a_i^\dagger a_i$, which in turn, (from Eq.9) implies that its energy will be given by

$$E = (N - 1)\omega - (L + \frac{\alpha}{2}N(N - 1))\omega, \tag{19}$$

- i.e., they are states with energies linearly falling with α. Similarly, any state annihilated by b_i will have an energy dependence that linearly rises with α -

$$E = (N - 1)\omega + (L + \frac{\alpha}{2}N(N - 1))\omega. \tag{20}$$

Thus Q_4^1 annihilates states with linearly rising energies, Q_4^2 annihilates states with linearly falling energies and Q_4^3 and Q_4^4 annihilate states with both linearly rising and falling energies. Moreover, we can also check that if $Q_4^i|\psi\rangle$ vanishes, then $[Q_4^i, K_+]|\psi\rangle$ also vanishes. Hence, the entire $S0(2, 1)$ family obtained by acting K_+ on the appropriate linear state or states is

annihilated by Q_4^i. But unlike the three anyon case, here all the states annihilated by Q_4^i need not be of the above form -*i.e.*, need not be linear states.

What does all this tell us about the spectrum ? For all states at statistics parameter α other than the linear states (and perhaps some non-linear states) that are annihilated by Q_4^i and the $\ell = 0$ states, there exist partner states at statistics $1 - \alpha$ with energies $E \pm \omega$ and angular momentum $\ell \pm 1$. Hence, given the spectrum at α, we can (almost) predict the spectrum at $1 - \alpha$. This is what we mean by saying that the problem of four anyons in a harmonic potential has a 'skewed' mirror symmetry. (This is similar to the symmetry found in the two anyon case [10].)

The more interesting case occurs when $N = 5$. Here, we can construct a unique operator

$$Q_5 = \epsilon_{ijkl} a_i^\dagger a_j b_k^\dagger b_l \tag{21}$$

which can be proven to be anti-symmetric under $z_m \leftrightarrow z_n$ by a straightforward extension of the argument that we used for the four anyon case. The determinant in Eq.16 is replaced by a 4×4 determinant β_{mnop}, which can be shown to be equal to $\epsilon_{mnop}/\sqrt{5}$, using the Jacobi coefficients in Eq.4. Moreover, it commutes with both the Hamiltonian and the angular momentum -

$$[H, Q_5] = 0 \quad \text{and} \quad [L, Q_5] = 0. \tag{22}$$

However, just as the Q_4^i, it is not a 'good' operator per se, because some terms in Q_5 give rise to singularities as $u_1 \to 0$, when acting on the $\ell = 0$ wavefunctions in (i). Q_5 is non-singular when acting on all other physical wavefunctions. Also, just as for the Q_4^i, we can show that Q_5 annihilates linear states, - states with energies of the form given in Eqs.19 and 20 because each term in Q_5 contains both a_i and b_i for some i. It is also easy to check that $[Q_5, K_+] = 0$. Hence, all members of the $SO(2, 1)$ family formed from the base linear states are annihilated by Q_5.

Hence, we conclude that *all* states except the $\ell = 0$ states and the linear states (and perhaps a few other non-linear states annihilated by Q_5) of the five anyon system come in pairs. Each state with energy E and angular momentum ℓ at a statistics parameter α is accompanied by a state with the same E and same ℓ at statistics parameter $1 - \alpha$. *In fact, we make the specific prediction that if the five anyon spectrum were computed and all the $\ell \neq 0$ non-linear states were plotted as a function of the statistics parameter α, then there would be an exact mirror symmetry about $\alpha = 1/2$.* Since, the $\ell \neq 0$ set of states is much larger than the $\ell = 0$ set of states, it would not be surprising if the fifth virial coefficient shows a mirror symmetry about $\alpha = 1/2$. However, to really demonstrate that, we need to show that the linear states (and any other states that may be annihilated by Q_5) and $\ell = 0$ states do not contribute to the difference in the virial coefficients $a_5(\alpha) - a_5(1 - \alpha)$. We have been unable to show this so far.

We note that this generalisation of Q_3 cannot be extended to cases beyond $N = 5$, (four relative coordinates) because there are only four types of oscillators, $a_i, b_i, a_i^\dagger$ and $b_i^\dagger$, and no further antisymmetrisation is possible. It may still be possible that for $N = 7, 9, \cdots$, the spectrum remains almost mirror symmetric, but other methods will have to be found to prove it.

In conclusion, in this paper, we have studied the symmetries of the few-anyon spectrum in some detail. We have constructed new fermionic operators (see Eqs.12 and 21). For the four

anyon problem, these operators act as raising and lowering operators of the Hamiltonian, but for the five anyon case, this fermionic operator commutes with the Hamiltonian. We have been able to show for the five anyon case that the system exhibits a mirror symmetry about $\alpha = 1/2$ (except for $\ell = 0$ and linear states). Thus, we have strengthened the conjecture [9] that all odd virial coefficients are mirror symmetric.

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